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A MATHEMATICAL MODEL FOR THE 3-DIMENSIONAL OCEAN SOUND PROPAGATION

DING LEE

Naval Underwater Systems Center
New London, CT 06320
U.S.A.

WILLIAM L. SIEGMANN

Rensselaer Polytechnic Institute
Troy, NY 12181
U.S.A.

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Abstract—A mathematical model is developed to represent sound propagation in a 3-dimensional ocean. The complete development is based on characteristics of the physical environment, mathematical theory, and computational accuracy. An exact solution test is performed to examine the validity of the theoretical development. A real application is included to demonstrate the model's capability.

1. INTRODUCTION

While the 2-dimensional underwater acoustic wave propagation problem is not yet completely solved for range-dependent environments, 3-dimensional environmental effects, such as fronts and eddies, certainly cannot be neglected. To predict underwater sound propagation, one usually deals with the solution of the Helmholtz equation, the reduced wave equation. This elliptic equation is well-defined with a set of boundary conditions, including the wall condition at the maximum range; this problem is purely a boundary value problem. An existing approach to economically solve this 3-dimensional range-dependent problem is by means of a 2-dimensional parabolic partial differential equation, originally introduced by Tappert[1]. This parabolic approximation approach, within the limitation of mathematical and acoustical approximations, offers efficient solutions to a class of long range propagation problems. The parabolic wave equation is much easier to solve than the elliptic equation; one big saving is the removal of the wall boundary condition at the maximum range. The application of the 2-dimensional parabolic wave equation to a number of realistic problems has been successful. Baer and Perkins[2, 3] extended the 2-dimensional parabolic wave equation to the 3-dimensional case in order to handle some 3-dimensional environmental effects. Pierce[4] also formulated a simplified 3-dimensional parabolic wave equation. Baer and Perkins[3] used the original Tappert development for their applications, which is limited to small-angle propagations. Pierce's formulation deals with the arc length, which is not easy to calculate.

This study begins with the discussion of the 3-dimensional elliptic wave equation in general and shows how to transform the elliptic equation into parabolic equations, which are easier to solve. The development in this study represents wide-angle propagation and

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accommodates previous developments by other authors as a special case. A few particularly important points will be illustrated. In the course of this development of the wide-angle 3-dimensional underwater acoustic wave equation, the physical properties, the mathematical validity, and computational accuracy are the main factors to be considered. We describe how the parabolic wave equation was derived and how the wide-angle propagation is taken into consideration. Then, a discussion on the limitations and the advantages of the parabolic equation approximation is highlighted. These discussions build up the background for the mathematical formulation of the 3-dimensional underwater acoustic wave propagation model. An exact solution test is given to examine the validity of the theoretical development. In addition, a real application is included to demonstrate the model's capability.

2. BACKGROUND OF THE 2-DIMENSIONAL PARABOLIC WAVE EQUATION

The introduction of the Parabolic Equation approximation (PE) by Tapper[1] to solve the reduced wave equation for a class of underwater acoustic wave propagation problems is recognized as an important simplification of a complicated ocean acoustic problem. However, most of the work on the PE is directed toward the solution of the 2-dimensional problem. The existing algorithms include the Split-step Fourier[5], the Implicit Finite-Difference (IFD)[6, 7], and Ordinary-Differential-Equation methods[8, 9]. The Split-step algorithm solves the PE efficiently for initial value problems while other methods can be used to solve the PE for initial-boundary value problems. Implementation of these solutions into computer codes has been made by Brock[10], Jensen and Krol[11], and Lee and Botseas[7]. The numerical results produced in this study were performed by the IFD package[7].

2.1. *Conventional development*

In order to give the reader an understanding of the parabolic equation approximation, we give a brief review of the mathematical formulation below.

Let $k(r, \theta, z)$ be the wave number which is a function of range, azimuthal angle, and depth. The representative underwater wave equation, for a harmonic point source, can be expressed in cylindrical coordinates (r, θ, z) as

$$\nabla^2\phi + k^2(r, \theta, z)\phi = 0, \tag{1}$$

where ∇^2 is the Laplacian operator and $\phi(r, \theta, z)$ is the complex acoustic pressure field. For more details, the reader is advised to consult Tolstoy and Clay[12].

Assuming slow variation in the azimuthal direction, we neglect the angular derivative, so that Eq. (1) can be expressed as the 2-dimensional elliptic equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + k_0^2 n^2(r, z)\phi = 0, \tag{2}$$

where k_0 is a reference wave number and $n(r, z)$ is the index of refraction. They are related to the wave number $k(r, z)$ by $k(r, z) = (2\pi f/c_0)\,(c_0/c(r, z)) = k_0 n(r, z)$, where c_0 is the reference sound speed, $c(r, z)$ is the sound speed, and f is the source frequency.

The parabolic equation approximation expresses

$$\phi(r, z) = u(r, z)v(r), \tag{3}$$

where $v(r)$ has strong dependence on the range variable r while $u(r, z)$ is weakly dependent on r. Substituting (3) into (2), we obtain

$$\left[v_{rr} + \frac{1}{r} v_r\right] u + \left[u_{rr} + \left(\frac{1}{r} + \frac{2}{v} v_r\right) u_r + u_{zz} + k_0^2 n^2(r, z) u\right] v = 0. \quad (4)$$

We can require that the first [ ] of (4) equals to $-k_0^2 v$ and the second [ ] of (4) equals to $k_0^2 u$; we find

$$v_{rr} + \frac{1}{r} v_r + k_0^2 v = 0, \quad (5)$$

and

$$u_{rr} + \left(\frac{1}{r} + \frac{2}{v} v_r\right) u_r + u_{zz} + k_0^2 (n^2(r, z) - 1) u = 0. \quad (6)$$

The solution of Eq. (5) for the outgoing wave in the range direction is the zeroth order Hankel function of the first kind. By applying the far-field approximation, $k_0 r \gg 1$, to the argument of the Hankel function, we obtain

$$v(r) = H_0^{(1)}(k_0 r) \cong \sqrt{\frac{2}{\pi k_0 r}}\, e^{i\left(k_0 r - \frac{\pi}{4}\right)}. \quad (7)$$

Use (7) to simplify the coefficient $1/r + (2/v)\, v_r$ of Eq. (6), which becomes

$$u_{rr} + 2ik_0 u_r + u_{zz} + k_0^2 (n^2(r, z) - 1) u = 0. \quad (8)$$

A key approximation to transform Eq. (8) into a parabolic equation is to drop the u_{rr} term, which was justified by the paraxial approximation, $|u_{rr}| \ll |2ik_0 u_r|$. This is permitted for the reason that, if the main radial dependence of the acoustic field is $e^{ik_0 r}$ for some choice of k_0, then the envelope u will vary more slowly as a function of r.

The dropping of u_{rr} simplifies Eq. (8) to be

$$u_r = \frac{i}{2} k_0 (n^2(r, z) - 1) u + \frac{i}{2k_0} u_{zz}. \quad (9)$$

Equation (9) is the *Standard Small Angle PE* introduced by Tappert[1].

2.2. *A wide-angle approach*

Having obtaineed Eq. (8), let us proceed differently and apply a rational function approximation, first considered by Claerbout[13], to formulate a wide-angle PE. We shall indicate the necessary limitations with respect to the formulation.

We express Eq. (8) in an operator form as follows:

$$\left(\frac{\partial}{\partial r} + ik_0 - ik_0 \sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}}\right) \times \left(\frac{\partial}{\partial r} + ik_0 + ik_0 \sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}}\right) u = 0. \quad (10)$$

For convenience, write $x = (n^2(r, z) - 1) + (1/k_0^2)\, \partial^2/\partial z^2$. Equations (8) and (10) are equivalent if and only if

$$\frac{\partial}{\partial r} \sqrt{1 + x}\, u = \sqrt{1 + x}\, \frac{\partial}{\partial r} u. \tag{11}$$

To show when Eq. (11) holds, we use a rational function approximation of the square root operator $\sqrt{1 + x}$ and assume that the $n(r, z)$ is a slowly varying function in range. Thus, range derivatives of $n(r, z)$ are neglected. We obtain:

$$\sqrt{1 + x} = \frac{1 + px}{1 + qx} = 1 + (p - q)x - q(p - q)x^2 + 0(x^3)$$

$$\begin{aligned} \frac{\partial}{\partial r} \sqrt{1 + x}\, u &= \frac{\partial}{\partial r} u + (p - q)\, (n^2(r, z) - 1) \frac{\partial}{\partial r} u \\ &+ \frac{(p - q)}{k_0^2} \frac{\partial^3}{\partial r \partial z^2} u - q(p - q) \left(\frac{\partial}{\partial r} x^2\right) u. \end{aligned}$$

$$\begin{aligned} \sqrt{1 + x}\, \frac{\partial}{\partial r} u &= \frac{\partial}{\partial r} u + (p - q)\, (n^2(r, z) - 1) \frac{\partial}{\partial r} u \\ &+ \frac{(p - q)}{k_0^2} \frac{\partial^3}{\partial z^2 \partial r} u - q(p - q) \left(x^2 \frac{\partial}{\partial r}\right) u. \end{aligned}$$

Since $(\partial/\partial r)\cdot(\partial/\partial z)\, u$ is assumed continuous, then $(\partial/\partial r)\, (\partial/\partial z)\, u = (\partial/\partial z)\, (\partial/\partial r)\, u$. Therefore, Eq. (10) is equivalent to Eq. (8) under these conditions.

Consider only the outgoing wave, from Eq. (10), we have

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p \left[n^2(r, z) - 1 + \dfrac{1}{k_0^2} \dfrac{\partial^2}{\partial z^2} \right]}{1 + q \left[n^2(r, z) - 1 + \dfrac{1}{k_0^2} \dfrac{\partial^2}{\partial z^2} \right]} \right) u. \tag{12}$$

If $q = 0$, $p = \frac{1}{2}$, it can be verified easily that Eq. (12) reduces to Eq. (9), the standard small-angle PE.

The reason that Eq. (9) is regarded as a small-angle PE is because the operator x can be shown to be estimated by $\sin^2 \theta$, where θ defines the angle of propagation. Hence, $\sqrt{1 + x} = 1 + (\frac{1}{2})x + 0(x^2)$ is valid for small-angle θ. It follows that the rational function approximation, which can be arranged to fit $\sqrt{1 + x}$ to second order in x by proper choice of p and q, is valid for larger-angle θ.

2.3. *A wide-angle propagation example*

To demonstrate the effect of propagation angle, we review the following example[14]:

This example has a range-independent environment which consists of an isovelocity water column and an isovelocity half-space bottom. Both the source and the receiver are placed at the same depth, 90.5 m below the surface. The source frequency is 250 Hz. The sound speed profile is given in Fig. 1 below. The sound speed is 1500 m/s in the water column and 1590 m/s in the bottom. There is a density change from 1.0 gm/cm^3 in the water column to 1.2 gm/cm^3 in the bottom. There is no attenuation in the water, but there

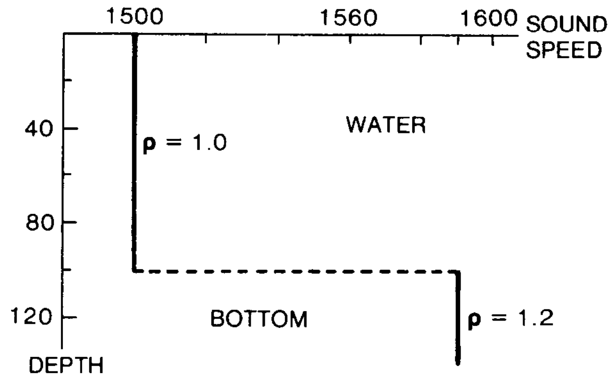

Fig. 1. Sound speed profile.

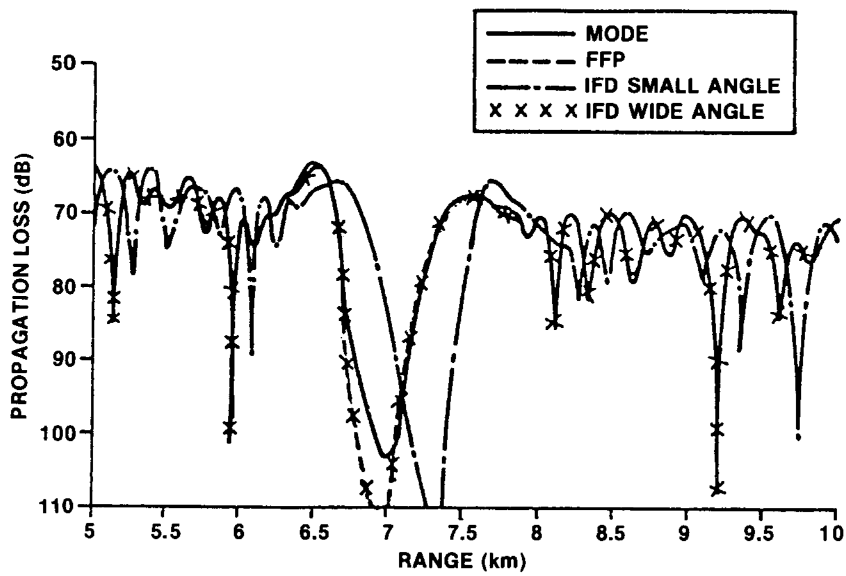

Fig. 2. Wide-angle solution comparison.

is an attenuation of 0.5 dB/λ in the bottom. The propagation loss was calculated up to a maximum range of 10 km. The maximum propagation angle for this problem is calculated to be approximately 19°; therefore, the wide-angle capability is expected to be important.

An exact solution, obtained by the fast field method[14], is used as a benchmark reference solution. As shown by the dashed line in Fig. 2, without the wide-angle capability, a phase error is evident; however, with the wide-angle capability, the IFD model (shown by x symbols in Fig. 2) gives excellent agreement with the reference solution.

This example demonstrates the importance of a wide-angle capability in shallow water underwater acoustic propagation.

3. BASIC LIMITATIONS AND ADVANTAGES OF THE PARABOLIC EQUATION APPROXIMATION

Bear in mind that the wide angle 2-dimensional PE, Eq. (12), is derived under the following limitations:

1. Far-field approximation: $k_0 r \gg 1$,
2. One way outgoing wave,
3. $n(r, z)$ is slowly varying in the range variable r,
4. A rational function approximation for the square root operator to shift the angular spread of propagation.

The above limitations arise in going from the elliptic equation to the parabolic equation. In solving the elliptic equation by means of the PE approximation, while obeying these limitations, the PE can be shown to have some important advantages. Let us compare the process of solution of the PE against that of the Helmholtz equation.

Consider a boundary value problem for the Helmholtz equation as follows:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + k_0^2 n^2(r, z)\phi = 0 \tag{13}$$

and the associated boundary conditions (see Fig. 3a):

$$\text{Initial boundary condition: } \phi(r_0, z) = \phi_I(z) \tag{13a}$$

$$\text{Surface boundary condition: } \phi(r, 0) = \phi_0(r) \tag{13b}$$

$$\text{Bottom boundary condition: } \phi(r, z_B) = \phi_B(r) \tag{13c}$$

$$\text{Wall boundary condition: } \phi(r_W, z) = \phi_W(z) \tag{13d}$$

in the region $0 \le z \le z_B$, $r_0 \le r \le r_W < \infty$.

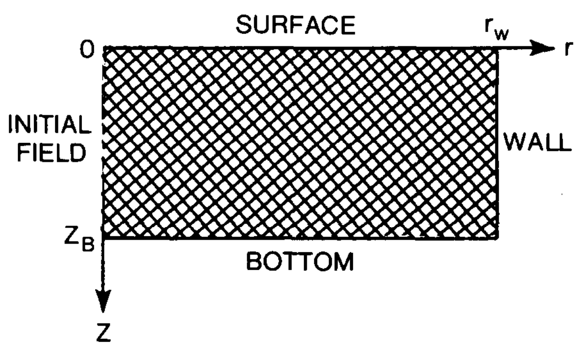

Fig. 3a.

Next, we formulate problem (13) with associated conditions (13a) through (13d) by means of the parabolic equation approximation. The related parabolic equation takes the expression

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p\left[(n^2(r, z) - 1) + \dfrac{1}{k_0^2}\dfrac{\partial^2}{\partial z^2}\right]}{1 + q\left[(n^2(r, z) - 1) + \dfrac{1}{k_0^2}\dfrac{\partial^2}{\partial z^2}\right]} \right) u. \tag{14}$$

The associated conditions are

$$\text{Initial condition: } u(r_0, z) = u_1(z) \tag{14a}$$

$$\text{Surface condition: } u(r, 0) = u_0(r) \tag{14b}$$

$$\text{Bottom condition: } u(r, z_B) = u_B(r) \tag{14c}$$

in the same region as above (see Fig. 3b). The region of parabolic propagation becomes:

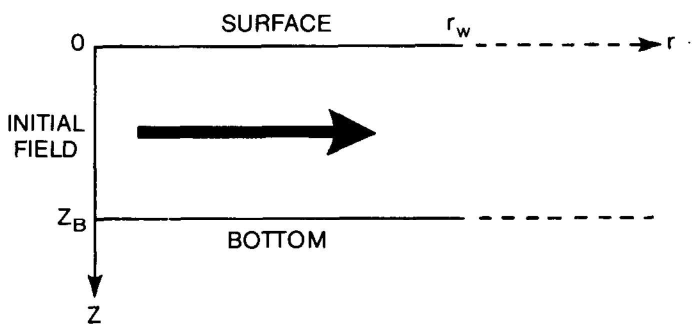

Fig. 3b.

Numerically solving Eq. (13) usually results in a large, sparse system of equations. The solution is considered accurate if the wall condition can be supplied accurately. The solution is considered inexpensive if the wall condition can be derived economically. On the other hand, using the PE method, the problem is solved by marching in range. This marching procedure avoids the large memory storage requirement. Furthermore, the need for the wall boundary condition is completely eliminated.

We have given a brief background for the 2-dimensional problem. We can now proceed to formulate mathematically a 3-dimensional parabolic wave propagation model.

4. FORMULATION OF THE WIDE ANGLE 3-DIMENSIONAL PE

We begin with the 3-dimensional Helmholtz equation in cylindrical coordinates, which is the analog of Eq. (2):

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + k_0^2 n^2(r, \theta, z)\phi = 0. \tag{15}$$

Equation (15) is solved subject to the following conditions:

$$\text{Initial condition: } \phi(r_0, \theta, z) = \phi_0(\theta, z) \tag{15a}$$

$$\text{Surface condition: } \phi(r, \theta, 0) = \phi_s(r, \theta) \tag{15b}$$

$$\text{Bottom condition: } \phi(r, \theta, z_B) = \phi_B(r, \theta) \tag{15c}$$

and

$$\text{Wall condition: } \phi(r_w, \theta, z) = \phi_w(\theta, z) \tag{15d}$$

Azimuthally, $\phi(r, \theta, z)$ might be taken as periodic, so that

$$\phi(r, \theta + 2\pi, z) = \phi(r, \theta, z). \tag{15e}$$

Following the development of the 2-dimensional PE, we let

$$\phi(r, \theta, z) = u(r, \theta, z)v(r), \tag{16}$$

where $v(r)$ again is assumed strongly dependent on r, and $u(r, \theta, z)$ is only weakly dependent on r. Substituting (16) into (15), we find

$$\left[v_{rr} + \frac{1}{r}v_r\right]u + \left[u_{rr} + \left(\frac{1}{r} + \frac{2}{v}v_r\right)u_r + u_{zz} + \frac{1}{r^2}u_{\theta\theta} + k_0^2 n^2(r,\theta,z)u\right]v = 0. \tag{17}$$

Again, set the first [ ] of Eq. (17) equal to $-k_0^2 v$ and set the second [ ] of Eq. (17) equal to $k_0^2 u$ to obtain

$$v_{rr} + \frac{1}{r}v_r + k_0^2 v = 0, \tag{18}$$

and

$$u_{rr} + \left(\frac{1}{r} + \frac{2}{v}v_r\right)u_r + u_{zz} + \frac{1}{r^2}u_{\theta\theta} + k_0^2(n^2(r, \theta, z) - 1)u = 0. \tag{19}$$

The far-field approximation and the consideration of only a one-way outgoing wave enable us to use the asymptotic expansion of the solution $v(r)$ of Eq. (18), i.e.,

$$v(r) = H_0^{(1)}(k_0 r) \cong \sqrt{\frac{2}{\pi k_0 r}}\, e^{i\left(k_0 r - \frac{\pi}{4}\right)}. \tag{20}$$

Using Eq. (20) to simplify the coefficient $(1/r + (2/v)\, v_r)$ in Eq. (19), we obtain a partial differential equation which is:

$$u_{rr} + 2ik_0u_r + u_{zz} + \frac{1}{r^2}u_{\theta\theta} + k_0^2(n^2(r, \theta, z) - 1)u = 0. \tag{21}$$

We name Eq. (21) the *Transformed Wave Equation*.

If we justify the neglect of u_{rr} in Eq. (21) for the same reason we used in the derivation of the 2-dimensional small angle PE Eq. (9), we obtain the small angle 3-dimensional PE, considered by Tappert-Baer-Perkins[3], which is

$$u_r = \frac{i}{2}k_0(n^2(r, \theta, z) - 1)u + \frac{i}{2k_0}u_{zz} + \frac{i}{2k_0}\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}. \tag{22}$$

Expressing Eq. (21) in operator form, we find

$$\left(\frac{\partial}{\partial r} + ik_0 - ik_0\sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial\theta^2}}\right)$$
$$\times\left(\frac{\partial}{\partial r} + ik_0 + ik_0\sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial\theta^2}}\right)u = 0. \tag{23}$$

We also assign a name to Eq. (23), the *Operator Wave Equation.*

In general, the *Transformed Wave Equation* and the *Operator Wave Equation* are not the same. In the following paragraph we discuss conditions so that these two wave equations are effectively equivalent, subject to the satisfaction of these properties.

The first operator in Eq. (23) represents the outgoing wave and the second operator represents the incoming wave. Considering, as in the 2-dimensional case, only the outgoing wave, we obtain

$$\left(\frac{\partial}{\partial r} + ik_0 - ik_0 \sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}}\right) u = 0. \tag{24}$$

To interpret Eq. (24), we need to express exactly how to treat the square root. We choose the same type of rational function approximation as in the 2-dimensional case:

$$\sqrt{1 + x + y} = \frac{1 + p_1 x + p_2 y}{1 + q_1 x + q_2 y}, \tag{25}$$

where

$$x = (n^2(r, \theta, z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}, \tag{26}$$

$$y = \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}, \tag{27}$$

and p_1, p_2, q_1, q_2 are constants which influence the size of the vertical propagation angle. For one example, a generalization of the Claerbout[13] development, these coefficients have the value $p_1 = p_2 = \frac{3}{4}$, $q_1 = q_2 = \frac{1}{4}$, which insure an approximation of the square root to second order in x and y. In general, Eq. (24) becomes

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p_1\left[(n^2(r, \theta, z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] + p_2 \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}}{1 + q_1\left[(n^2(r, \theta, z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] + q_2 \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}} \right) u. \tag{28}$$

Equation (28) is termed the *Wide Angle 3-Dimensional Parabolic Equation.*

The terminology is deceptive, because Eq. (28) is not a parabolic equation. In fact, it is a third-order partial differential equation. Because the small angle parabolic equation is a special case and because of our accustomed terminology, we refer to it as a PE. Equation (12) is a 2-dimensional counterpart of Eq. (28), obtained by neglecting the terms involving $(1/r^2)\ \partial^2/\partial\theta^2$ and by considering $n(r, \theta, z)$ as azimuthally independent. If one carries out the multiplication of the two operators in Eq. (23), one can find that Eqs. (21) and (23) are not identical unless a condition is satisfied. This is certainly the situation found earlier for Eqs. (8) and (10), and the condition, which is analogous to that below Eq. (11), is:

$$\frac{\partial}{\partial r}\left(\sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial}{\partial \theta^2}}\right) u$$

$$= \left(\sqrt{1 + (n^2 - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}}\right) \frac{\partial}{\partial r} u. \tag{29}$$

Prior to indicating the justification of this condition, two important properties must be discussed. We recall that Eq. (21) is obtained from Eq. (19) by means of a far-field approximation applied to the solution $v(r)$. This far-field approximation, $k_0 r \gg 1$, enables the solution $v(r)$ of Eq. (18) to be expressed by Eq. (20). If we denote

$$\bar{v}(r) = \sqrt{\frac{2}{\pi k_0 r}}\, e^{i\left(k_0 r - \frac{\pi}{4}\right)}, \tag{30}$$

then a measure of the relative error of $v(r)$ is

$$E(v) = |\, v(r) - \bar{v}(r) \,|\, |\, \bar{v}(r) \,|^{-1}. \tag{31}$$

In computation, suppose we require

$$E(v) \le \delta, \tag{32}$$

where δ is some specified tolerance. The tolerance condition (32) focuses on difference in modulus, rather than differences in phase which are of less interest. It is known[15] that

$$|\, v(r) \,| \sim |\, \bar{v}(r) \,| \left\{ 1 - \frac{1}{16(k_0 r)^2} + 0\, \frac{1}{(k_0 r)^4} \right\},\; k_0 r \to \infty. \tag{33}$$

The terms in the curly brackets of Eq. (33) alternate in sign, and they have the property that the remainder after retaining any number of terms is no bigger than the first term neglected. From Eqs. (32) and (33), it follows that $\bar{v}(r)$ is regarded as acceptably approximating $v(r)$ if

$$k_0 r \ge 1/(4\sqrt{\delta}). \tag{34}$$

PROPERTY 1. The solution field $v(r)$ of Eqs. (7) or (20) is said to have Property 1 if condition (34) is satisfied for an arbitrarily assigned δ.

In terms of acoustic frequency f, condition (34) requires the range r to satisfy

$$r \ge r_f = c_0/(8\pi f \sqrt{\delta}) \tag{35}$$

in which r_f is the minimum range for the far-field approximation to apply. As an example, suppose $\delta = 0.01$, which corresponds to differences between $v(r)$ and $\bar{v}(r)$ being bounded by 1 percent. Then, as f increases from, say, 10 to 200 Hz, r_f decreases from 60 m to 3 m. An alternate mathematical expression for this case is $k_0 r \ge 2.5$. Now, we have established a lower bound for $k_0 r$ for the far-field approximation to be valid. This establishment enables us to next approximately justify the condition (29).

To define the square root of $(1 + x + y)$, we choose to use the rational representation, Eq. (25). For simplicity here, as employed in the 2-dimensional case, we extend the Claerbout[13] coefficients to the 3-dimensional case. That is, we use coefficients $p_1 = p_2 = \frac{3}{4}$, $q_1 = q_2 = \frac{1}{4}$.

For convenience, write

$$\sqrt{1 + x + y} = \sqrt{1 + Z} = \frac{1 + \frac{3}{4} Z}{1 + \frac{1}{4} Z}. \tag{36}$$

where

$$Z = x + y = (n^2(r, \theta, z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}. \tag{37}$$

The primary advantage of this approximation is that it is correct to the second order of Z:

$$\sqrt{1 + Z} = (1 + \tfrac{1}{4}Z)^{-1}(1 + \tfrac{3}{4}Z) \cong 1 + \tfrac{1}{2}Z - \tfrac{1}{8}Z^2 + 0(Z^3). \tag{38}$$

Note that

$$\begin{aligned} Z^2 = {} & (n^2(r, \theta, z) - 1)^2 + \frac{1}{k_0^4}\frac{\partial^4}{\partial z^4} + \frac{1}{(k_0 r)^4}\frac{\partial^4}{\partial \theta^4} \\ & + \frac{2}{k_0^4 r^2}\frac{\partial^4}{\partial \theta^2 \partial z^2} + \frac{(n^2(r, \theta, z) - 1)}{k_0^2}\left[\frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] \\ & + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\left[(n^2(r, \theta, z) - 1) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(n^2(r, \theta, z) - 1)\right]. \end{aligned} \tag{39}$$

It follows that effects of terms in Eq. (39) are included in Eq. (36). Even though Eq. (28) explicitly contains no fourth-order z and θ derivatives, the effects of fourth-order derivatives are in some sense incorporated properly in Eq. (28).

The condition of Eq. (29) states clearly that the factorization approximation, Eq. (23), of Eq. (21) is exact when $\sqrt{1 + Z}$ and $\partial/\partial r$ commute. Since this is not true in general, Eq. (23) can be expanded to yield Eq. (21) with the presence of the additional term

$$ik_0\left[\frac{\partial}{\partial r}\sqrt{1 + Z} - \sqrt{1 + Z}\frac{\partial}{\partial r}\right]u = \xi u. \tag{40}$$

In view of Eq. (38), it follows from Eq. (40),

$$\xi u \cong \xi^{(1)}u + \xi^{(2)}u, \tag{41}$$

where

$$\xi^{(1)} = \frac{ik_0}{2}\left[\frac{\partial}{\partial r}Z - Z\frac{\partial}{\partial r}\right], \tag{42}$$

and

$$\xi^{(2)} = -\frac{ik_0}{8}\left[\frac{\partial}{\partial r}Z^2 - Z^2\frac{\partial}{\partial r}\right]. \tag{43}$$

Using the Z expression of Eq. (37), we find that the leading term ξ has the form

$$\xi = \xi^{(1)} = ik_0 n(r, \theta, z)\frac{\partial n}{\partial r} - \frac{i}{k_0 r^3}\frac{\partial^2}{\partial \theta^2}. \tag{44}$$

A similar expression can be developed for the second term in ξ, as given by Eq. (43), but for simplicity here we focus on the leading term of Eq. (44). By arguments which are analogous to, but lengthier than, those used to derive Eq. (34), it can be shown[16] that Eq. (44) is negligible compared to other terms retained in Eq. (28) provided

$$\frac{\partial n}{\partial r} \leq 4\delta k_0(\max|\, n^2 - 1\,|\,) \tag{45}$$

and

$$k_0 r \geq 1/(4\sqrt{\delta}) = k_0 r_f/\sqrt{\delta}. \tag{46}$$

The equality in Eq. (46) involving r_f follows from the definition of r_f (See Eq. (35)).

PROPERTY 2. The solution field $u(r, \theta, z)$ of Eq. (28) is said to have Property 2 if conditions (45) and (46) are satisfied for an arbitrarily assigned δ.

The rational function approximation for $\sqrt{1 + Z}$ plus the satisfaction of Property 2 are *sufficient* to justify condition (29). Moreover, we observe that condition (34) is automatically satisfied by the stronger requirement (46). Hence, we have shown the following result:

The Transformed Wave Equation (21) and the Operator Wave Equation (23) are equivalent if the operator approximation (36) is appropriate and if the wave field possesses Property 2.

From this result we can, thus, deal with the Operator Wave Equation. We consider only the one-way outgoing wave and apply the rational function approximation for the associated square root operator. Based on our development, the wide angle 3-dimensional PE, Eq. (28), is introduced. The Property 1 defines a range of validity for the far-field approximation. Assuming slow variation in the index of refraction as in Eq. (45) and for r bigger than $\sqrt{\delta}\, r_f$, the wide angle 3-dimensional PE should be applied. As range increases to a point where $(1/k_0^2)\,(1/r^2)\,\partial^2 u/\partial\theta^2$ is small, then the wide angle 2-dimensional PE, Eq. (12), should be applied; in this case, for N θ-partitions, it is equivalent to solve a "$N \times 2D$" problem as proposed by Perkins and Baer[3].

5. NUMERICAL SOLUTION ALGORITHM

In pursuing the solution to the wide angle 3-dimensional PE, Eq. (28), we use the symbols x and y (see Eqs. (26) and (27)) for brevity, and deal with the equation in the form

$$\frac{\partial}{\partial r} u = \left(-ik_0 + ik_0 \frac{1 + p_1 x + p_2 y}{1 + q_1 x + q_2 y} \right) u. \tag{47}$$

An efficient numerical solution to Eq. (47) has been worked out by Schultz, Lee, and Jackson[17]. We summarize their solution below.

Writing

$$\mathcal{L} = -ik_0 + ik_0 \frac{1 + p_1 x + p_2 y}{1 + q_1 x + q_2 y},$$

then Eq. (47) can be expressed by

$$\frac{\partial}{\partial r} u = \mathscr{L} u. \tag{48}$$

A half-half splitting is used to express the solution $u(r, \theta, z)$ implicitly by

$$e^{-(1/2)\, k\partial/\partial r} u^{n+1} = e^{(1/2)\, k\partial/\partial r} u^{n} \tag{49}$$

where $k = \Delta r$.

The Crank-Nicolson scheme to solve Eq. (49) takes the expression

$$\left(1 - \frac{1}{2} k \frac{\partial}{\partial r}\right) u^{n+1} = \left(1 + \frac{1}{2} k \frac{\partial}{\partial r}\right) u^{n}. \tag{50}$$

Using the true operator expression for $\partial u/\partial r$, central differences for $\partial^2 u/\partial z^2$ and $\partial^2 u/\partial \theta^2$, and simplifying, we obtain the following system of equations:

$$\begin{aligned} P_{m,l} u_{m,l}^{n+1} &+ Q(u_{m+1,l}^{n+1} + u_{m-1,l}^{n+1}) + R(u_{m,l+1}^{n+1} + u_{m,l-1}^{n+1}) \\ &= P_{m,l}^{+} u_{m,l}^{n} + Q^{*}(u_{m+1,l}^{n} + u_{m-1,l}^{n}) + R^{+}(u_{m,l+1}^{n} + u_{m,l-1}^{n}) \end{aligned} \tag{51}$$

where

$$\begin{aligned} P_{m,l} = &\left(1 + q_1(n^2(r, \theta, z) - 1) - \frac{2q_1}{k_0^2}\frac{1}{h^2} - \frac{2q_2}{k_0^2(r+k)^2\delta^2}\right) \\ &- i\left(\frac{1}{2} k_0 k(p_1 - q_1)(n^2(r, \theta, z) - 1)\right. \\ &\left. - \frac{k}{k_0}\frac{1}{h^2}(p_1 - q_1) - \frac{k}{k_0}\frac{1}{(r+k)^2}\frac{1}{\delta^2}(p_2 - q_2)\right), \end{aligned}$$

$$Q = \frac{1}{k_0}\frac{1}{h^2}\frac{q_1}{k_0} - \frac{i}{2} k(p_1 - q_1),$$

$$R = \frac{1}{k_0}\frac{1}{(r+k)^2}\frac{1}{\delta^2}\frac{q_2}{k_0} - \frac{i}{2} k(p_2 - q_2),$$

$$\begin{aligned} P_{m,l}^{+} = &\left(1 + q_1(n^2(r, \theta, z) - 1) - \frac{2q_1}{k_0^2}\frac{1}{h^2} - \frac{2q_2}{k_0^2 r^2\delta^2}\right) \\ &+ i\left(\frac{1}{2} k_0 k(p_1 - q_1)(n^2(r, \theta, z) - 1)\right. \\ &\left. - \frac{k}{k_0}\frac{1}{h^2}(p_1 - q_1) - \frac{k}{k_0}\frac{1}{r^2}\frac{1}{\delta^2}(p_2 - q_2)\right), \end{aligned}$$

$$R^{+} = \frac{1}{k_0}\frac{1}{r^2}\frac{1}{\delta^2}\left(\frac{q_2}{k_0} + \frac{i}{2} k(p_2 - q_2)\right),$$

m is the index of depth increment, l is the index of θ increment, n is the index of range increment, and $h = \Delta z$, $k = \Delta r$, $\delta = \Delta\theta$. In general, $m = 1, 2, \ldots, M$ and $l = 1, 2,$

. . . , L. The system (51) to be solved is in the form:

$$A\,u^{n+1} = B\,u^{n} + u_0^{n+1} + u_0^{n}, \tag{52}$$

where u_0^j contain boundary information in both the present range level and the next range level, and A and B are matrices both possessing the format.

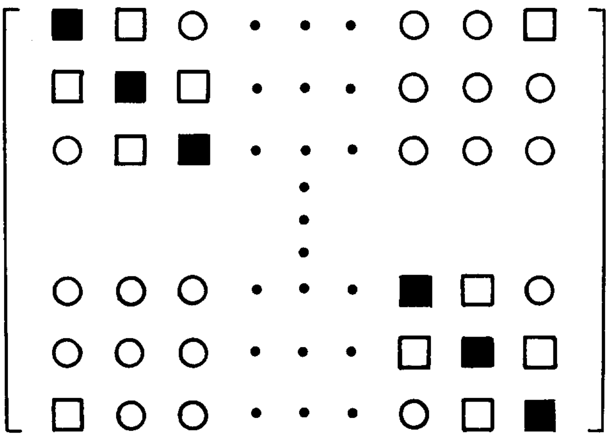

Fig. 4.

The shaded block matrices are all tri-diagonal with diagonal elements $P_{m,l}$, off diagonal elements Q, and the blank block matrices which are all diagonal with elements R. Elements of matrix A are $P_{m,l}$, Q, and R. Elements of matrix B are $P_{m,l}^{+}$, Q^{*}, and R^{+}. System (51) is a symmetric, complex, large sparse system with seven diagonals. The Conjugate Gradient method was applied to solve the system. This method is a portion of the Yale Sparse package[18]. Computational results are presented in the example section.

Attention should be paid to the fact that proceeding from Eq. (50) to the formulation of Eq. (51), a very lengthy derivation is involved. The complete development has been worked out by Schultz, Lee, and Jackson[17].

6. NUMERICAL RESULTS

To gain a true feeling of this wide angle 3-dimensional PE model, we present two examples in this section. One is using an exact solution to test the validity of this model. The other is using a known reference solution to examine the capability in an actual application.

6.1. *An exact solution test*

To carry out the numerical test, we use the Yale University Sparse Technique[18] to solve Eq. (51) for both small and wide angle cases. We partition the azimuthal plane into ten sectors, each sector being 36° wide. We use specific initial values and associated boundary conditions, which are described later in this section. We output the solution at the boundary of each sector so that it can be compared with a known reference result. We now describe this convenient reference solution.

We construct a solution $u(r, \theta, z)$ as a product of three functions, each being a function of a single variable (r, θ, or z). We require this solution to satisfy Eq. (28) for arbitrary p_1, p_2, q_1, and q_2. To meet this requirement, we rewrite Eq. (28) in the following form:

$$\left\{1 + q_1(n^2(r, \theta, z) - 1) + \frac{q_1}{k_0^2}\frac{\partial^2}{\partial z^2} + \frac{q_2}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}\right\}\frac{\partial}{\partial r}u$$
$$= ik_0\left\{(p_1 - q_1)\left[(n^2(r, \theta, z) - 1) + \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2}\right] + (p_2 + q_2)\frac{1}{(k_0 r)^2}\frac{\partial^2}{\partial \theta^2}\right\}u. \quad (53)$$

We look for a solution $u(r, \theta, z)$ such that

$$u(r, \theta, z) = \sin(\Omega z)\, e^{im\theta}\phi(r). \quad (54)$$

Substituting Eq. (54) into Eq. (53), we find

$$\left\{1 + q_1\left[(n^2(r, \theta, z) - 1) - \frac{\Omega^2}{k_0^2}\right] - \frac{q_2 m^2}{k_0^2}\frac{1}{r^2}\right\}\phi_r$$
$$= ik_0\left\{(p_1 - q_1)\left[(n^2(r, \theta, z) - 1) - \frac{\Omega^2}{k_0^2}\right] - \frac{(p_2 - q_2)m^2}{k_0^2}\frac{1}{r^2}\right\}\phi. \quad (55)$$

For computational simplicity, we choose $n^2(r, \theta, z) - 1 - \Omega^2/k_0^2 = 0$. Since $k = k_0 n(r, \theta, z) = \omega/c$, then,

$$k_0 = \left[\left(\frac{\omega}{c}\right)^2 - \Omega^2\right]^{1/2}. \quad (56)$$

The choice of k_0, as given by formula (56), enables Eq. (55) to be written in a simple form, viz,

$$\frac{d\phi}{dr} = \left(\frac{-ik_0(p_2 - q_2)m^2/(k_0 r)^2}{1 - q_2 m^2/(k_0 r)^2}\right)\phi = -if(r)\phi, \quad (57)$$

which is a first order differential equation

$$\phi' + if(r)\phi = 0. \quad (58)$$

The solution to Eq. (58) is:

$$\phi = Ae^{-i\int f(r)\,dr}, \quad (59)$$

and the $\int f(r)\,dr$ may be found for given parameter values.

Before a discussion of particular cases, we note the initial and boundary conditions satisfied by our solution. The initial values are taken from

$$u(r_0, \theta, z) = \sin(z)e^{im\theta}\phi(r_0). \quad (60)$$

Possible associated boundary conditions are:

$$\text{Surface:}\quad u(r, \theta, z_0) = \sin(z_0)e^{im\theta}\phi(r) = 0 \quad (61)$$

$$\text{Bottom:}\quad u(r, \theta, z_B) = \sin(z_B)e^{im\theta}\phi(r) = 0. \quad (62)$$

These boundary conditions are satisfied by the choices

$$\Omega z_0 = 0 \text{ and } \Omega z_B = \text{an integer multiple of } \pi.$$

In our numerical examples, we choose the integer multiple as 2. The initial range is selected as $r_0 = 50$ m so that the far-field approximation (see Eq. (35)) is satisfied for frequency of interest.

In the computations, the numerical results represent the solution wave field, expressed as the Propagation Loss (PL), as a ratio of two intensities in decibel (dB) units. This PL calculation is performed by means of the conventional formula below.

$$\text{PL} = -20 \log_{10} | u(r, \theta, z) | \tag{63}$$

(i) *Case* 1: Small-angle propagation ($p_1 = p_2 = 1/2$, $q_1 = q_2 = 0$). An evaluation of $\int f(r)\, dr$ gives $-\, m^2/2k_0 r$, which leads to the solution

$$\phi(r) = Ae^{i}\, m^2/2k_0 r. \tag{64}$$

The numerical results are presented in Table 1 at every depth z, where the first row indicates the computed results, and the second row indicates the exact solution Eqs. (54) and (64). The results are taken from the fourth azimuthal sector at 108° at range 50.4 m, with a value of $m = 3$. The depths z are in meters, corresponding to a choice of $z_B = 200$ meters. The selection of these parameters has no particular physical significance but demonstrates the accuracy of the computation. We note the closeness in the solution $u(r, \theta, z)$ and PL values between the exact and calculated solutions.

(ii) *Case* 2. Wide-angle propagation ($p_1 = p_2 = \frac{3}{4}$, $q_1 = q_2 = \frac{1}{4}$). An evaluation of $\int f(r)\, dr$ gives the solution

$$\phi(r) = Ae\left(\frac{-im(p_2 - q_2)}{2\sqrt{q_2}} \ln\left[\frac{k_0 r - m\sqrt{q_2}}{k_0 r + m\sqrt{q_2}}\right]\right). \tag{65}$$

Numerical results are compared against the exact solution are given in Table 2 in the

Table 1. Results of small-angle propagation

z	$PL\ (DB)$	u	
30.00	12.636	(0.18834E + 00	−0.13793E + 00)
30.00	12.636	(0.18886E + 00	−0.13722E + 00)
60.00	6.859	(0.36627E + 00	−0.26824E + 00)
60.00	6.859	(0.36729E + 00	−0.26685E + 00)
90.00	3.749	(0.52397E + 00	−0.38372E + 00)
90.00	3.749	(0.52541E + 00	−0.38174E + 00)
120.00	1.841	(0.65270E + 00	−0.47800E + 00)
120.00	1.841	(0.65451E + 00	−0.47553E + 00)
150.00	0.688	(0.74537E + 00	−0.54587E + 00)
150.00	0.688	(0.74743E + 00	−0.54304E + 00)
180.00	0.108	(0.79685E + 00	−0.58357E + 00)
180.00	0.108	(0.79906E + 00	−0.58055E + 00)

Table 2. Results of wide-angle propagation

z	PL (DB)	u	
30.00	12.645	(0.22578E + 00	−0.58389E + 00)
30.00	12.636	(0.22613E + 00	−0.57994E + 00)
60.00	6.868	(0.43904E + 00	−0.11362E + 00)
60.00	6.859	(0.43976E + 00	−0.11278E + 00)
90.00	3.757	(0.62819E + 00	−0.16245E + 00)
90.00	3.749	(0.62909E + 00	−0.16134E + 00)
120.00	1.852	(0.78225E + 00	−0.20236E + 00)
120.00	1.841	(0.78365E + 00	−0.20098E + 00)
150.00	0.694	(0.89377E + 00	−0.23150E + 00)
150.00	0.688	(0.89492E + 00	−0.22952E + 00)
180.00	0.117	(0.95544E + 00	−0.24602E + 00)
180.00	0.108	(0.95673E + 00	−0.24537E + 00)

same formats as Table 1. Once again, note the excellent agreement between exact and computed solutions.

6.2. *Discussion*

It is important to observe the behavior of the solution of Case 2 for large k_0r. An examination of the real part of the solution gives, for

$$\alpha = \frac{m(p_2 - q_2)}{2\sqrt{q}} \ln\left[\frac{k_0r - m\sqrt{q_2}}{k_0r + m\sqrt{q_2}}\right],$$

$$\cos(\alpha) = \cos\left\{\frac{m(p_2 - q_2)}{2\sqrt{q_2}}\left[\ln\left(1 - \frac{m\sqrt{q_2}}{k_0r}\right) - \ln\left(1 + \frac{m\sqrt{q_2}}{k\delta}\right)\right]\right\}$$

$$= \cos\left\{\frac{m(p_2 - q_2)}{2\sqrt{q_2}}\left[\left\{-\frac{m\sqrt{q_2}}{k_0r} - \frac{1}{2}\frac{m^2q_2}{(k_0r)^2} - \cdots\right\}\right.\right.$$

$$\left.\left. - \left\{\frac{m\sqrt{q_2}}{k_0r} - \frac{1}{2}\frac{m^2q_2}{(k_0r)^2} + \cdots\right\}\right]\right\}$$

For k_0r large,

$$\cos(\alpha) \approx \cos\left\{\frac{m(p_2 - q_2)}{2\sqrt{q_2}}\left[-\frac{2\sqrt{q_2}m}{k_0r}\right]\right\} = \cos\left\{-\frac{m^2(p_2 - q_2)}{k_0r}\right\}. \tag{66}$$

For large k_0r and for wide-angle parameters $p_2 - q_2 = 1/2$, Eq. (66) reduces to the real part of the solution of Case 1, as it must. Furthermore, in examining the limiting function $\cos(m^2/2k_0r)$, we note that this function increases monotonically after $k_0r = m^2/\pi$ and approaches unity as $k_0r \to \infty$. When this function is close to unity, 3-dimensional effects are "lost," and Eq. (28) behaves like the 2-dimensional parabolic wave equation, Eq. (14). In particular, for $k_0r \geq 10m^2/\pi$, this function differs from unity by less than 1 percent, so that only 2-dimensional propagation is seen.

Finally, we remark that the more rapid the azimuthal variation (i.e., the bigger m^2),

Table 3. A Pacific profile

z (m)	$c(z)$ (m/s)
0.00	1536.500
152.40	1539.243
406.30	1501.143
1015.90	1471.882
5587.91	1549.606
>5587.91	1555.526

the further out in range do 3-dimensional effects influence the solution. This result is expected on physical grounds.

6.3. *An application*

The same numerical procedures are carried out for this application as for the exact solution test. The sound speed profile (see Table 3) has a large linear gradient in the cross range direction. As described in [2] and [19], the gradient is chosen as 1 m/s per km. The profile can be described mathematically by the formula $c(r, \theta, z) = c_m(z) + (0.001)\, r \sin \theta$ where $c_m(z)$ takes on the values in the table below. This type of phenomenon can be seen across the Gulf Stream. This profile, in the vertical plane at 0°, is characterized as a typical profile in the North Pacific Ocean.

The source is placed at 254 m below the surface with a frequency of 25 Hz, and the receiver is placed at 815 m. The propagation is carried out up to 140 km in range. Again, we produced results in sectors. A plot describing the section at angle 0° is given in Fig. 5.

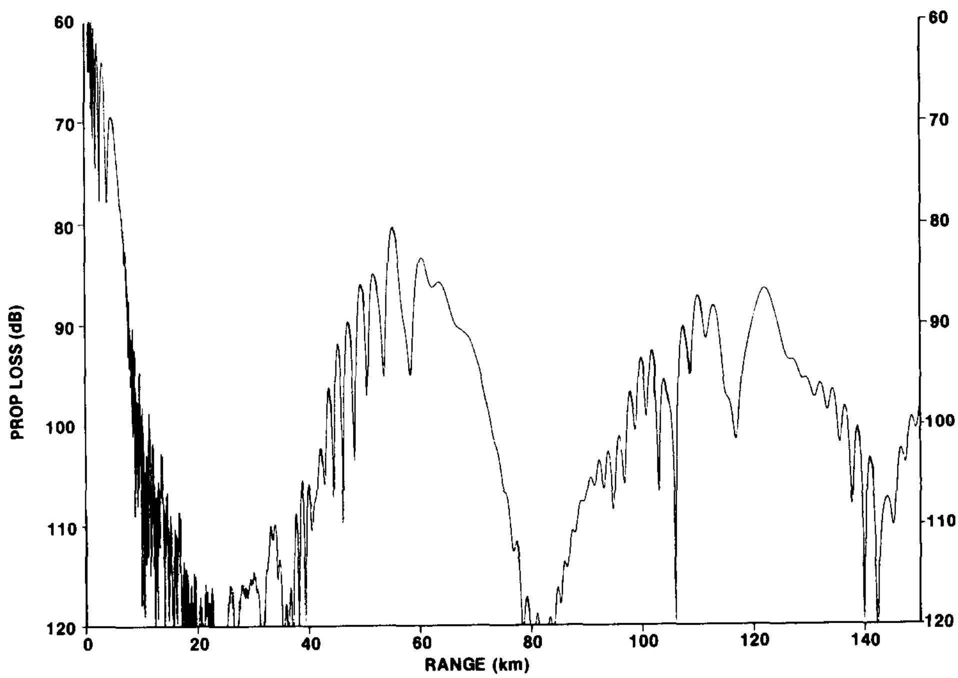

Fig. 5. Propagation loss vs. range.

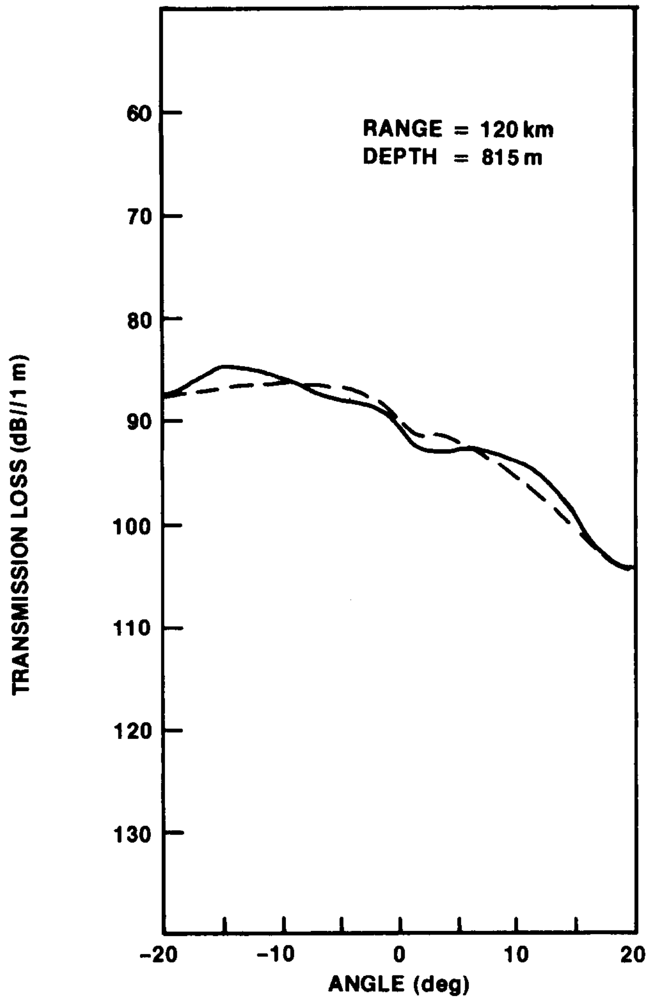

Fig. 6. Propagation loss as a function of azimuthal angle between $-20°$ and $20°$ at a range 120 km and a depth of 815 m.

Of interest is the reading from Fig. 5 at 120 km, of approximately 90 dB. A comparison with the reference solution[2], presented in Fig. 6, shows that the PL reading at 120 km is also approximately 90 dB. In Fig. 6, the dotted line is a 2-dimensional solution and the solid line is a 3-dimensional solution both obtained with the split-step Fourier algorithm[2]. The dotted solution is computed at each θ value by the so-called "$N \times 2D$" algorithm[3]. Good agreement with a known reference solution in this application, plus the agreement with the exact solution test, indicates that the computational accuracy of this 3-dimensional model is satisfactory.

7. CONCLUSIONS

Modeling the mathematical solution to 3-dimensional underwater acoustic wave propagation problems not only involves difficulties in describing the theoretical acoustics but also involves complications due to large-scale computations. We have made use of the actual mathematical and physical properties of the problem to simplify a great deal. These simplified developments allow us to introduce a 3-dimensional mathematical model to predict underwater acoustic wave propagation. Our wide angle 3-dimensional PE model is theoretically justifiable and computationally accurate. In conjunction with the application of advanced Yale Sparse Matrix Techniques, this model offers a variety of capabilities to handle a class of long-range propagation problems under realistic acoustical environments.

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